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On the characteristic lengths for absorbing, emitting, and scattering media

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Abstract—This work presents a comprehensive development of characteristic lengths for multidimensional, absorbing, emitting, and scattering media. The length representing emission phenomena is discussed in detail while the length representing the reflection and transmission contributions is simply derived. The characteristic lengths are developed from the photon path length approach, yet the lengths are calculable without knowledge of the photon path length (PPL) distribution functions. Limiting values and the relation to the mean beam length are presented. Numerical results for planar and cylindrical media for ranges of optical depth and scattering albedo are presented and discussed.

1. INTRODUCTION

THE CHARACTERISTIC length of a physical system often gives insight into basic phenomena of the problem. The concept is used extensively in scaling analyses in many engineering disciplines. In radiative transfer the mean beam length and geometric mean beam length are rigorously defined characteristic lengths which serve to scale a complex volume to a single line-of-sight. The mean beam length is defined as the radius of a gas hemisphere that emits a flux to its base center point equal to the average flux emitted by the actual gas volume to the area of concern. The mean beam length essentially reduces the complete volume integration to a single line-of-sight calculation and is very useful in computing the emission of isothermal gas volumes of many shapes. The geometric mean beam length generalizes the mean beam length so that radiation exchange between surface pairs of an isothermal gas volume and real gas behavior are included. In this case, the shape factor expressions are integrated in the linear gas absorption regime so that band and growth models for real gas emission are incorporated. Tabulated values of the length are then required for each set of surface configurations.

While the above lengths characterize emission from isothermal gas volumes, neither includes the influence of scattering in its definition. Scattering redirects the paths of photon travel and renders useless the area and volume integrations possible for non-scattering media. For reflection and transmission problems in non-emitting media, the mean photon path lengths have been defined [1–3] to incorporate the effect of absorption and the redistribution of photon path lengths that occurs as a result of scattering. The calculation of this characteristic length requires only the radiative heat flux and the derivative of the heat flux in the absorbing and scattering (non-emitting) medium and is therefore quite general. However, the reflection/transmission mean photon path length does not characterize medium behavior when volumetric emission is allowed. One attempt to do so has been presented by Cartigny [4] who studied the effect of a weakly scattering component on the mean beam length for the case of volume emission by an optically thin medium.

This work therefore introduces characteristic lengths that are of interest in multidimensional, absorbing, scattering, and emitting media. The photon path length (PPL) method of solving for the energy transfer is used to develop the lengths. Once developed, however, the characteristic lengths are calculable independent of any knowledge of the PPL distribution functions.

2. PHOTON PATH LENGTH ANALYSIS

The medium of concern in this study is illustrated in Fig. 1 and consists of an absorbing, scattering, and emitting medium that is homogeneous and isothermal with temperature T. Scattering is assumed to be coherent but may be anisotropic. The volume's shape is arbitrary with the proviso that the outer boundary cannot view itself (i.e. the boundary is convex). The outer boundary is assumed to be composed of a number of diffuse differential area elements dA_s, each with its own surface properties (area, temperature or heat flux, and emissivity). The values of these properties are functions of the location $\mathbf{r}_{dA_s}$ of dA_s on the boundary. The analysis which follows is done on a spectral basis, and the absorption and scattering within the medium are denoted by an absorption coefficient a_ν and scattering coefficient σ_ν where ν denotes wave number. The medium phase function is denoted by P_ν.

Consider a differential area element dA located at

NOMENCLATURE

a_ν	spectral absorption coefficient [m^{-1}]
a'_ν	secondary spectral absorption coefficient [m^{-1}]
A_s	total surface area of the boundary [m^2]
$dq^\pm_{\nu rt}$	spectral flux at dA from redirected/transmitted radiosity of surface element dA_s [W cm m^{-2}]
dA	differential area element at which the flux is evaluated [m^2]
dA_s	differential surface area element [m^2]
dV	differential volume element [m^3]
D	cylinder diameter [m]
$i_{\nu b}$	spectral blackbody intensity [W cm m^{-2} sr^{-1}]
$i_{\nu so}$	outgoing spectral intensity of surface element dA_s [W cm m^{-2} sr^{-1}]
l	photon path length [m]
$\langle l\rangle^\pm_\nu$	mean length of photon travel for volume emitted energy arriving at dA [m]
$\langle l\rangle^\pm_{\nu e}$	mean emission length for volume emitted energy arriving at dA [m]
$\langle l\rangle^\pm_{\nu rt}$	mean length of photon travel for redirected/transmitted energy leaving dA_s and arriving at dA [m]
$\langle l\rangle^\pm_{\nu}$	additional mean length for volume emitted energy arriving at dA [m]
L	total layer depth [m]
L_{mb}	surface area average mean beam length, $4V/A_s$ [m]
$L_{mb,l}$	local mean beam length [m]
$\mathbf{n}_{dA}$	normal vector to area element dA
P_ν	spectral phase function
$q^\pm_e$	total flux at dA from volume emission [W m^{-2}]
$q^\pm_\nu$	spectral flux at dA from volume emission and redirected/transmitted surface radiosity [W cm m^{-2}]
$q^\pm_{\nu e}$	spectral flux at dA from volume emission [W cm m^{-2}]
r	radial coordinate of the cylinder [m]
$\mathbf{r}_{dA}$	location of area element dA
$\mathbf{r}_{dA_s}$	location of surface area element dA_s
$\mathbf{r}_{dV}$	location of volume element dV
T	medium temperature [K]
V	total medium volume [m^3]
y	layer coordinate normal to the boundaries [m]
z	axial coordinate of the cylinder [m].

Greek symbols

β_ν	spectral extinction coefficient, $(a_\nu+\sigma_\nu)$ [m^{-1}]
ε'_ν	directional, spectral emission quantity
ν	wave number [cm^{-1}]
σ_ν	spectral scattering coefficient [m^{-1}]
$\Phi^\pm(l,\mathbf{r}_{dA_s})$	path length function for redirected/transmitted energy [sr m^{-3}]
$\Psi^\pm(l)$	path length function for volume emission [sr]
ω_ν	scattering albedo, $\sigma_\nu/(a_\nu+\sigma_\nu)$.

Superscripts

+	positive direction
−	negative direction
′	secondary quantity, directional quantity.

Subscripts

b	blackbody
e	emission quantity
l	local quantity
mb	mean beam length quantity
o	outgoing
rt	redirected/transmitted quantity
s	surface quantity
ν	wave number dependent quantity.

$\mathbf{r}_{dA}$, internal to the volume V, and let dA have positive (+) and negative (−) directions relative to a defined surface normal $\mathbf{n}_{dA}$. At element dA, the energy flux in either the (+) or (−) direction, $q^\pm_\nu$, is composed of the flux that results from diffuse surface radiosity of dA_s that is redirected and transmitted to dA in a non-emitting medium, $dq^\pm_{\nu rt}$, and the flux resulting from volume emitted energy from all volume elements dV which reaches dA after absorption and scattering in the medium with a transparent boundary, $q^\pm_{\nu e}$. Note that for both $q^\pm_{\nu e}$ and $dq^\pm_{\nu rt}$, the entire medium is considered since scattering is a volume, rather than line-of-sight, phenomenon. The radiative energy flux at dA in either direction (±) from all surface sources dA_s and from all volume sources dV is

$$q^\pm_\nu(\text{surface properties}, T, a_\nu, \sigma_\nu, P_\nu, \mathbf{r}_{dA}) = \int_{A_s} dq^\pm_{\nu rt}(\text{surface properties}, \mathbf{r}_{dA_s}, a_\nu, \sigma_\nu, P_\nu, \mathbf{r}_{dA}) + q^\pm_{\nu e}(T, a_\nu, \sigma_\nu, P_\nu, \mathbf{r}_{dA}). \quad (1)$$

Explicit functional notation is included in equation (1) for completeness but will be eliminated in the remaining equations. The net energy flux at dA is then

$$q_\nu = q^+_\nu - q^-_\nu. \quad (2)$$

In developing an expression for $dq^\pm_{\nu rt}$, it is necessary to define $\Phi^\pm(l, \mathbf{r}_{dA_s})\,dl$ as the PPL probability function for photons with travel lengths between l and $l+dl$

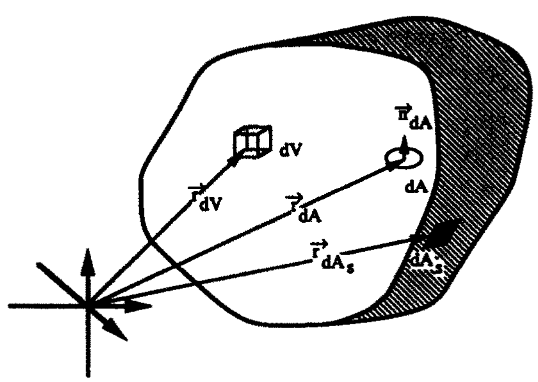

FIG. 1. Multidimensional, isothermal, homogeneous medium with a convex boundary.

that originate from a source of strength $\pi\,\mathrm{d}A_s$ (a diffuse source of unit intensity and area $\mathrm{d}A_s$) and reach $\mathrm{d}A$ in either the (+) or (−) direction in a conservative medium. For the conservatively scattering medium ($a_v = 0$) and unit intensity source

$$\mathrm{d}q^{\pm}_{\mathrm{vrt}} = \mathrm{d}A_s \int_0^{\infty} \Phi^{\pm}(l, \mathbf{r}_{\mathrm{d}A_s})\,\mathrm{d}l. \tag{3}$$

$\Phi^{\pm}(l, \mathbf{r}_{\mathrm{d}A_s})\,\mathrm{d}l$ represents the contribution to the redirected and transmitted flux at $\mathrm{d}A$ from photons originating at $\mathrm{d}A_s$ and having lengths between l and $l+\mathrm{d}l$ in a conservative medium. With a surface source of non-unit outgoing intensity, $\mathrm{d}q^{\pm}_{\mathrm{vrt}}$ becomes

$$\mathrm{d}q^{\pm}_{\mathrm{vrt}} = i_{\mathrm{vso}}\,\mathrm{d}A_s \int_0^{\infty} \Phi^{\pm}(l, \mathbf{r}_{\mathrm{d}A_s})\,\mathrm{d}l \tag{4}$$

where i_{vso} is the diffuse intensity leaving the surface. Finally, allowing the absorption coefficient a_v to be nonzero, $\mathrm{d}q^{\pm}_{\mathrm{vrt}}$ can be written as

$$\mathrm{d}q^{\pm}_{\mathrm{vrt}} = i_{\mathrm{vso}}\,\mathrm{d}A_s \int_0^{\infty} \mathrm{e}^{-a_v l}\,\Phi^{\pm}(l, \mathbf{r}_{\mathrm{d}A_s})\,\mathrm{d}l. \tag{5}$$

Note that the above equations are written in terms of the radiosity of surface element $\mathrm{d}A_s$ since a general enclosure analysis requires such a form for the surface exchange. Since the purpose of the current derivation is the development of characteristic lengths, a complete enclosure solution is not attempted here. Consequently, it is important to note that the lengths that are eventually developed are not dependent upon the source strength of $\mathrm{d}A_s$ and thus are valid for the exchange between $\mathrm{d}A$ and any diffuse area element $\mathrm{d}A_s$.

In formulating $q^{\pm}_{\mathrm{ve}}$, a method similar to the above is used. Let $\Psi^{\pm}(l, \mathbf{r}_{\mathrm{d}V})\,\mathrm{d}l$ denote the PPL probability distribution function for photons with travel lengths between l and $l+\mathrm{d}l$, which have been emitted by a volumetric source $\mathrm{d}V$ located at $\mathbf{r}_{\mathrm{d}V}$ and have arrived at area element $\mathrm{d}A$ in either the (+) or (−) direction in a conservative medium with a transparent boundary. The PPL function for the entire homogeneous volume $\Psi^{\pm}(l)$ is obtained by integrating over all elemental volumes

$$\Psi^{\pm}(l) = \int_V \Psi^{\pm}(l, \mathbf{r}_{\mathrm{d}V})\,\mathrm{d}V. \tag{6}$$

The internal sources in this case are considered to be diffuse thermal sources of strength $4\pi\,\mathrm{d}V$. The flux at $\mathrm{d}A$ which results from volume emitted energy that reaches $\mathrm{d}A$ in a conservative medium with sources of strength $4\pi a_v i_{vb}(T)\,\mathrm{d}V$ is

$$q^{\pm}_{\mathrm{ve}} = a_v i_{vb} \int_0^{\infty} \Psi^{\pm}(l)\,\mathrm{d}l. \tag{7}$$

$\Psi^{\pm}(l)\,\mathrm{d}l$ then represents the contribution to the emission flux at $\mathrm{d}A$ from photons with lengths between l and $l+\mathrm{d}l$. Introducing absorption, the emitted energy flux is

$$q^{\pm}_{\mathrm{ve}} = a_v i_{vb}(T) \int_0^{\infty} \mathrm{e}^{-a_v l}\,\Psi^{\pm}(l)\,\mathrm{d}l. \tag{8}$$

Noting that a directional spectral emission quantity for path length l can be written

$$\varepsilon'_v(l) = 1 - \mathrm{e}^{-a_v l} \tag{9}$$

and

$$\frac{\partial \varepsilon'_v(l)}{\partial l} = a_v\,\mathrm{e}^{-a_v l} \tag{10}$$

it is possible to rewrite equation (8) as

$$q^{\pm}_{\mathrm{ve}} = i_{vb}(T) \int_0^{\infty} \frac{\partial \varepsilon'_v(l)}{\partial l}\,\Psi^{\pm}(l)\,\mathrm{d}l. \tag{11}$$

Finally, integrating equation (11) by parts to introduce $\varepsilon'_v(l)$ from equation (9) rather than its derivative, $q^{\pm}_{\mathrm{ve}}$ reduces to

$$q^{\pm}_{\mathrm{ve}} = -i_{vb}(T) \int_0^{\infty} \varepsilon'_v(l)\,\frac{\partial \Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l. \tag{12}$$

Comparing equations (11) and (12), $\varepsilon'_v(l)$ in equation (12) describes an integrated or cumulative contribution from an entire PPL. On the other hand, $\partial\varepsilon'_v(l)/\partial l$ in equation (11) represents a more localized contribution from an emission source that has been transmitted by the absorbing constituent. Ultimately, the development of a characteristic length for emission will be of concern. Since the concept of emission as an integrated quantity rather than a local one is needed in radiative transfer analyses and experiments, the form of equation (12) is expected to be of more practical use than equation (11) for the derivation of a characteristic length.

With $\mathrm{d}q^{\pm}_{\mathrm{vrt}}$ and $q^{\pm}_{\mathrm{ve}}$ defined, the energy flux at $\mathrm{d}A$ is written using equations (1), (5), (9), and (12)

$$q^{\pm}_v = \int_{A_s} i_{\mathrm{vso}} \int_0^{\infty} \mathrm{e}^{-a_v l}\,\Phi^{\pm}(l, \mathbf{r}_{\mathrm{d}A_s})\,\mathrm{d}l\,\mathrm{d}A_s$$
$$- i_{vb}(T) \int_0^{\infty} (1 - \mathrm{e}^{-a_v l})\,\frac{\partial \Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l. \tag{13}$$

Then, using $\Psi^{\pm}(\infty) = 0$, equation (13) reduces to

$$q_v^{\pm} = \int_{A_s} i_{vso} \int_0^{\infty} e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl\, dA_s + i_{vb}(T)\Psi^{\pm}(0) + i_{vb}(T) \int_0^{\infty} e^{-a_v l} \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl. \quad (14)$$

In equation (14), $\Psi^{\pm}(0) = \pi$. To develop this result, consider a sphere of differential radius dl about a differential area element dA. Volumetric emission is isotropic and the product $a_v i_{vb}(T)$ is given the value of unity ($a_v i_{vb}(T) = 1$) in developing the function $\Psi^{\pm}(l)$. Therefore, the intensity at dA is diffuse and has the value $i_v = dl$. Integrating over each hemisphere to obtain the flux incident at dA

$$dq_{ve}^{\pm} = \pi\, dl. \quad (15)$$

Thus, for the area element dA, the energy incident on it from volume emission with length 0 to dl must have arrived from non-scattered energy emitted within a distance dl of the element. (Scattering can be assumed negligible over a differential length dl.) $\Psi^{\pm}(l)$ in the region of $l = 0$ can be defined as the volume emitted energy flux at dA in the (+) or (−) direction from photons with length 0 to dl divided by the length interval dl

$$\Psi^{\pm}_{0\text{-}dl} = \frac{dq_{ve}^{\pm}}{dl} = \pi. \quad (16)$$

Since dl can be made arbitrarily small (as can dA), it is valid to state that

$$\Psi^{\pm}(0) = \pi. \quad (17)$$

Finally, $q_v^{\pm}$ can be written as

$$q_v^{\pm} = \int_{A_s} i_{vso} \int_0^{\infty} e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl\, dA_s + \left[\pi i_{vb}(T) + i_{vb}(T) \int_0^{\infty} e^{-a_v l} \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl\right] \quad (18)$$

where the emission contribution (the term in brackets in equation (18)) is also denoted by

$$q_{ve}^{\pm} = \pi i_{vb}(T) + i_{vb}(T) \int_0^{\infty} e^{-a_v l} \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl. \quad (19)$$

This is a general expression for the radiative energy flux at dA. The above analysis can be directly extended to obtain the radiative intensities at dA.

3. CHARACTERISTIC LENGTHS

The energy flux $q_v^{\pm}$ involves two integrals over PPL. The first of these quantifies the amount of energy leaving dA_s that is redirected and transmitted by the medium to dA. The second integral involves a quantity useful in describing volume emission to dA by the medium. The fact that both integrals are over path length with the integrand composed of a probability function makes it natural to define characteristic mean lengths using the first moment of the integrand. Thus, a characteristic length for redirected and transmitted energy $\langle l \rangle_{vrt}^{\pm}$ is defined as the following mean:

$$\langle l \rangle_{vrt}^{\pm} = \frac{\int_0^{\infty} l\, e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl}{\int_0^{\infty} e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl} \quad (20)$$

which describes a mean length of travel for photons leaving dA_s and arriving at dA. Likewise, using the second integral, an emission length $\langle l \rangle_{ve}^{\pm}$ is defined as

$$\langle l \rangle_{ve}^{\pm} = \frac{\int_0^{\infty} l\, e^{-a_v l} \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl}{\int_0^{\infty} e^{-a_v l} \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl} \quad (21)$$

which describes a characteristic emission length for the volume radiating to dA. These two lengths are sufficient to characterize the radiative transfer phenomena included in $q_v^{\pm}$. However, for the sake of completeness, several other lengths which present themselves in the analysis are provided here without discussion. Returning to equation (8), a length which describes the mean length of photon travel for photons emitted volumetrically and arriving at dA can be defined as

$$\langle l \rangle_v^{\pm} = \frac{\int_0^{\infty} l\, e^{-a_v l}\, \Psi^{\pm}(l)\, dl}{\int_0^{\infty} e^{-a_v l}\, \Psi^{\pm}(l)\, dl}. \quad (22)$$

Equation (12) is used to introduce a second length as

$$\langle l \rangle_{\varepsilon}^{\pm} = \frac{\int_0^{\infty} l \varepsilon_v'(l) \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl}{\int_0^{\infty} \varepsilon_v'(l) \frac{\partial \Psi^{\pm}(l)}{\partial l}\, dl}. \quad (23)$$

Another length can be defined by viewing equation (21) in the limit of vanishing absorption coefficient ($a_v \to 0$). This length is independent of a_v.

The two characteristic lengths $\langle l \rangle_{vrt}^{\pm}$ and $\langle l \rangle_{ve}^{\pm}$ that are defined above are not of direct use in their present form since they require knowledge of the PPL distribution functions $\Phi^{\pm}(l, \mathbf{r}_{dA_s})$ and $\partial \Psi^{\pm}(l)/\partial l$. However, it is possible to circumvent this difficulty. For $\langle l \rangle_{vrt}^{\pm}$, consider equation (5)

$$dq_{vrt}^{\pm} = i_{vso}\, dA_s \int_0^{\infty} e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl.$$

Taking the derivative with respect to a_v of this equation, the following expression is obtained:

$$\frac{\partial (dq_{vrt}^{\pm})}{\partial a_v} = -i_{vso}\, dA_s \int_0^{\infty} l\, e^{-a_v l}\, \Phi^{\pm}(l, \mathbf{r}_{dA_s})\, dl. \quad (24)$$

Combining equations (5), (20), and (24), $\langle l \rangle_{vrt}^{\pm}$ is expressible as

$$\langle l\rangle_{\nu rt}^{\pm} = \frac{-1}{\mathrm{d}q_{\nu rt}^{\pm}}\frac{\partial(\mathrm{d}q_{\nu rt}^{\pm})}{\partial a_\nu} = -\frac{\partial[\ln(\mathrm{d}q_{\nu rt}^{\pm})]}{\partial a_\nu}. \quad (25)$$

Following the same procedure for $\langle l\rangle_{\nu e}^{\pm}$, consider equation (19)

$$q_{\nu e}^{\pm} = \pi i_{\nu b}(T) + i_{\nu b}(T)\int_0^\infty \mathrm{e}^{-a_\nu l}\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l.$$

The characteristic length is obtained with

$$\frac{\partial q_{\nu e}^{\pm}}{\partial a_\nu} = -i_{\nu b}(T)\int_0^\infty l\,\mathrm{e}^{-a_\nu l}\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l \quad (26)$$

to yield

$$\langle l\rangle_{\nu e}^{\pm} = \frac{-1}{[q_{\nu e}^{\pm} - \pi i_{\nu b}(T)]}\frac{\partial q_{\nu e}^{\pm}}{\partial a_\nu} = -\frac{\partial[\ln(\pi i_{\nu b}(T) - q_{\nu e}^{\pm})]}{\partial a_\nu}. \quad (27)$$

As an aside, note that it is possible to define second and higher-order moments, $\langle l^n\rangle_{\nu rt}^{\pm}$ and $\langle l^n\rangle_{\nu e}^{\pm}$ $(n \geqslant 2)$, using the same techniques applied in developing equations (25) and (27). In this case, higher-order derivatives of $\mathrm{d}q_{\nu rt}^{\pm}$ and $q_{\nu e}^{\pm}$ are necessary.

The significance of the forms of $\langle l\rangle_{\nu rt}^{\pm}$ and $\langle l\rangle_{\nu e}^{\pm}$ in equations (25) and (27) is that only $\mathrm{d}q_{\nu rt}^{\pm}$ and $q_{\nu e}^{\pm}$ and their derivatives are involved. That is, knowledge of or calculations involving the PPL analysis are unnecessary. With the derivatives calculated using a finite difference approximation, needed values of $\mathrm{d}q_{\nu rt}^{\pm}$ and $q_{\nu e}^{\pm}$ may be computed using any method desired for a particular problem–spherical harmonics, discrete ordinates, Monte Carlo, etc. Therefore, while PPL concepts are used to define the mean lengths, the concepts are unnecessary for the practical computation of $\langle l\rangle_{\nu rt}^{\pm}$ and $\langle l\rangle_{\nu e}^{\pm}$. In the following sections, $\langle l\rangle_{\nu e}^{\pm}$ will be discussed in greater detail. An extensive discussion of the value of $\langle l\rangle_{\nu rt}^{\pm}$ is provided in ref. [1] for the layer geometry, and additional exposition here is not warranted.

4. AN APPLICATION OF $\langle l\rangle_{\nu e}^{\pm}$

Up to this point, no mention has been made of how the characteristic length $\langle l\rangle_{\nu e}^{\pm}$ might be used in a radiative transfer problem. In order to provide an example of its application, consider the medium of Fig. 1 which is assumed to have constant scattering coefficient σ_ν and absorption coefficient a_ν. Now, introduce a secondary pure absorber into the volume with absorption coefficient a'_ν. The volume emitted energy flux at $\mathrm{d}A$, $q_{\nu e}^{\pm}$, is then identical to that given in equation (19) with $(a_\nu + a'_\nu)$ replacing a_ν

$$q_{\nu e}^{\pm} = \pi i_{\nu b}(T) + i_{\nu b}(T)\int_0^\infty \exp(-a'_\nu l) \times \exp(-a_\nu l)\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l. \quad (28)$$

Replacing $\exp(-a'_\nu l)$ by $[1-(1-\exp(-a'_\nu l))]$, equation (28) becomes

$$q_{\nu e}^{\pm} = \pi i_{\nu b}(T) + i_{\nu b}(T)\int_0^\infty \exp(-a_\nu l) \times\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l - i_{\nu b}(T)\int_0^\infty [1-\exp(-a'_\nu l)] \times\exp(-a_\nu l)\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l \quad (29)$$

$$q_{\nu e}^{\pm} = q_{\nu e}^{\pm}(a'_\nu = 0) - i_{\nu b}(T)\int_0^\infty [1-\exp(-a'_\nu l)] \times\exp(-a_\nu l)\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l. \quad (30)$$

Now, consider the limit of small $a'_\nu l$ (optically thin secondary absorber)

$$\lim_{a'_\nu l\to 0}[1-\exp(-a'_\nu l)] = \lim_{a'_\nu l\to 0}\left[a'_\nu l - \frac{(a'_\nu l)^2}{2} + \cdots\right] \cong a'_\nu l.$$

For small $a'_\nu l$, it is then possible to substitute equation (31) into equation (30) to obtain

$$q_{\nu e}^{\pm} \cong q_{\nu e}^{\pm}(a'_\nu = 0) - i_{\nu b}(T)a'_\nu \times\int_0^\infty l\exp(-a_\nu l)\frac{\partial\Psi^{\pm}(l)}{\partial l}\,\mathrm{d}l. \quad (32)$$

Introducing the characteristic emission length $\langle l\rangle_{\nu e}^{\pm}$ from equation (21), the energy transfer $q_{\nu e}^{\pm}$ becomes

$$q_{\nu e}^{\pm} \cong q_{\nu e}^{\pm}(a'_\nu = 0) - a'_\nu\langle l\rangle_{\nu e}^{\pm}[q_{\nu e}^{\pm}(a'_\nu = 0) - \pi i_{\nu b}(T)]. \quad (33)$$

Thus, $q_{\nu e}^{\pm}(a'_\nu > 0)$ can be calculated very easily for small $a'_\nu l$ using only $q_{\nu e}^{\pm}(a'_\nu = 0)$ and $\langle l\rangle_{\nu e}^{\pm}$. If a second-order finite difference approximation is used in calculating the derivative in equation (27), at most three calculations of $q_{\nu e}^{\pm}$ are necessary to completely define $q_{\nu e}^{\pm}(a'_\nu = 0)$ and $\langle l\rangle_{\nu e}^{\pm}$. Then, for given values of σ_ν and a_ν, $q_{\nu e}^{\pm}(a'_\nu > 0)$ may be computed approximately for any desired value of the secondary absorption coefficient a'_ν.

The methodology used in developing equation (33) is equally valid for the solution of the total energy transfer $q_e^{\pm}$ when the secondary absorber is more complex. For example, consider a medium composed of one constituent with constant absorption and scattering coefficients and a secondary, purely absorbing constituent with spectrally dependent absorption coefficient. If this secondary constituent is a real gas such as CO_2 or H_2O, integration over wave number introduces the band absorption for each of the gas bands present. The weak band limit of band absorption being linearly dependent upon path length l can be used to introduce $\langle l\rangle_{\nu e}^{\pm}$. This approach is presented for reflection and transmission problems with scat-

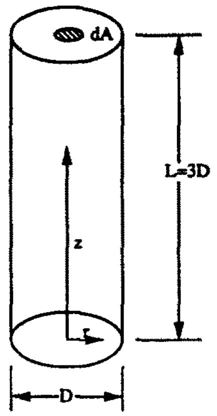

FIG. 2. Cylinder with aspect ratio $D/L = 1/3$.

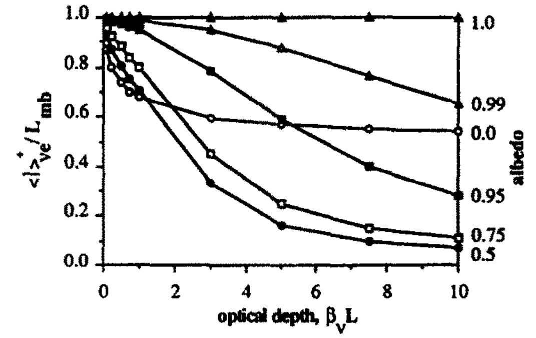

FIG. 3. Mean emission length at the surface of the planar layer.

tering media in ref. [1] and for emission problems with scattering media in ref. [5]. In these references, an optically thin secondary absorber result is demonstrated to apply to problems with higher optical depths.

5. LIMITING VALUES AND RESULTS FOR $\langle l \rangle^{\pm}_{ve}$

At this point, it is interesting to consider the behavior of $\langle l \rangle^{\pm}_{ve}$ in various limits and for various geometries. If $\langle l \rangle^{\pm}_{ve}$ is indeed a characteristic emission length, its expected value for an area element dA on the boundary of an optically thin medium ($\sigma_v \to 0$, $a_v \to 0$) is the local value of the mean beam length. This trend as well as the general behavior of $\langle l \rangle^{\pm}_{ve}$ with changing values of the absorption and scattering coefficients is discussed below.

To check the optically thin limit, first consider the geometry of a one-dimensional layer with a coordinate y normal to the layer boundaries and total depth L. In this case, the area element dA is placed at the boundary $y = L$ with the surface normal in the positive y-direction and $\langle l \rangle^{+}_{ve}$ at dA the length of interest. Then q^{+}_{ve} at dA for a layer with $\sigma_v = 0$ is

$$q^{+}_{ve} = [1 - 2E_3(a_v L)]\pi i_{vb}(T) \quad (34)$$

where $E_3(\,)$ is the third exponential integral function. Equation (27) is used to show that $\langle l \rangle^{+}_{ve}$ reduces to

$$\langle l \rangle^{+}_{ve} = \frac{LE_2(a_v L)}{E_3(a_v L)}, \quad \sigma_v = 0 \quad (35)$$

where $E_2(\,)$ is the second exponential integral function. Letting $a_v \to 0$ yields

$$\langle l \rangle^{+}_{ve} = 2L = L_{mb}, \quad a_v \to 0, \quad \sigma_v = 0 \quad (36)$$

where L_{mb} is the mean beam length for the layer volume radiating to its boundaries. Thus, the characteristic emission length approaches the mean beam length in the optically thin limit as expected. Next, consider a cylinder with an aspect ratio of $D/L = 1/3$ where D is the diameter and L the height (see Fig. 2).

Area dA is located at the center of the upper boundary with the surface normal in the positive z-direction. Again, $\langle l \rangle^{+}_{ve}$ at dA is the desired length. Assuming $\sigma_v = 0$ and $a_v \to 0$, a numerical evaluation of the PPL function $\partial \Psi^{+}(l)/\partial l$ is used in equation (21) to yield $\langle l \rangle^{+}_{ve}/L \cong 0.31$ which is equivalent to $L_{mb,l}/L$ for this geometry. $L_{mb,l}$ is the local value of the mean beam length as opposed to the surface area average value which is denoted simply as L_{mb}.

For more general values of the absorption coefficient a_v and scattering coefficient σ_v, a boundary value of the characteristic emission length $\langle l \rangle^{+}_{ve}$ exhibits interesting trends and differs greatly from the mean beam length. To illustrate this, consider the identical layer and cylinder geometries used for the above optically thin results. In both cases, $\langle l \rangle^{+}_{ve}$ at dA, calculable from equation (21) or (27), is desired for a range of extinction coefficient, $\beta_v = a_v + \sigma_v$, and scattering albedo, $\omega_v = \sigma_v/(a_v + \sigma_v)$. Scattering is assumed to be isotropic for all results. The results for the layer are given in Fig. 3 where $\langle l \rangle^{+}_{ve}/L_{mb}$ is plotted vs optical depth, $\beta_v L$, for several values of albedo. The cylinder results are provided in Figs. 4–11 where $\langle l \rangle^{+}_{ve}/L_{mb}$ is plotted vs extinction coefficient for the same values of albedo used in Fig. 3 and for various radial (r) and axial (z) positions of the cylinder boundary. The mean beam lengths used for the layer and cylinder results are the surface area average values, $L_{mb} = 4V/A_s$, where V is the medium volume and A_s

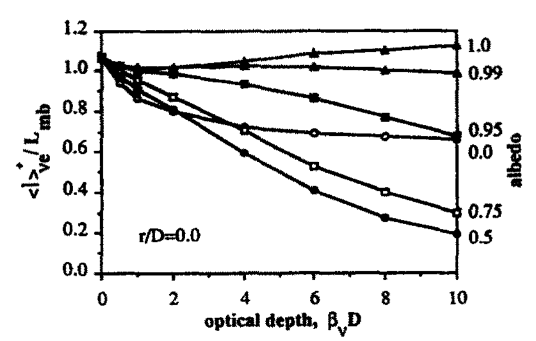

FIG. 4. Mean emission length at the surface of the cylinder at a radial location of $r/D = 0$.

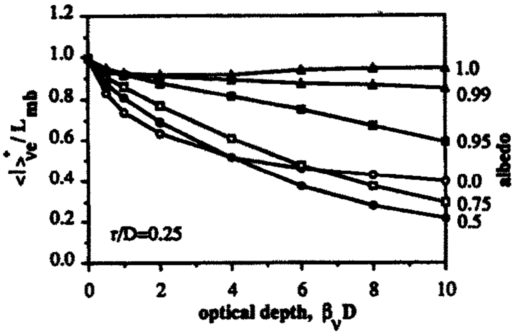

FIG. 5. Mean emission length at the surface of the cylinder at a radial location of $r/D = 0.25$.

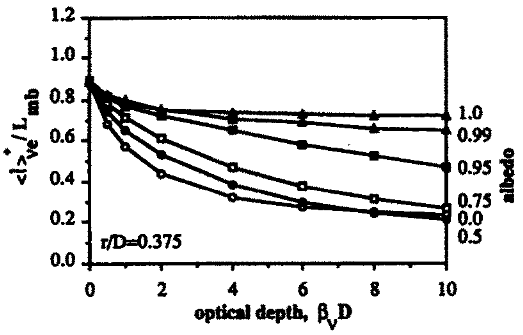

FIG. 6. Mean emission length at the surface of the cylinder at a radial location of $r/D = 0.375$.

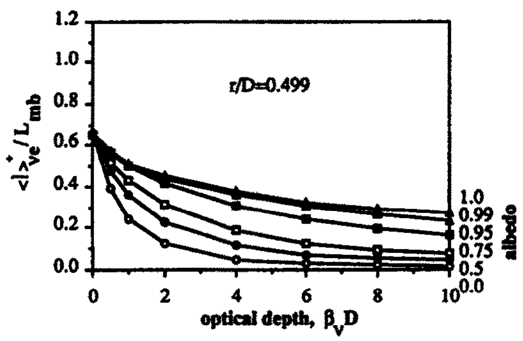

FIG. 7. Mean emission length at the surface of the cylinder at a radial location of $r/D = 0.499$.

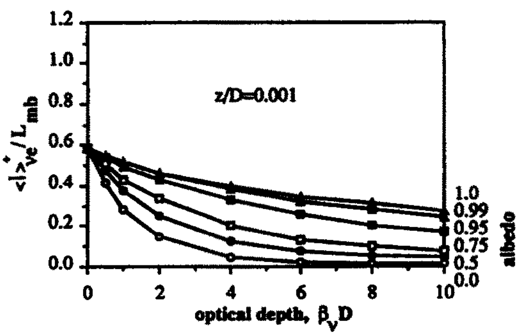

FIG. 8. Mean emission length at the surface of the cylinder at an axial location of $z/D = 0.001$.

the medium surface area. For the layer, the local and average values are identical.

The numerical results presented are obtained using a Monte Carlo simulation of the PPL analysis. The solution involves solving for the emission path length function $-\partial\Psi^{\pm}(l)/\partial l$ in the conservative medium with a transparent boundary after first noting an equivalence between this function and the path length function for diffuse surface incidence at a differential area element. This equivalence is demonstrated using conservation of energy arguments for diffuse incidence at a transparent surface element to obtain an emission path length formulation and then applying Laplace transform properties. Diffuse incidence at a boundary element is simulated via a Monte Carlo technique by following incident energy bundles through the scattering medium until the bundles reach the boundary. The total travel lengths are then used to compose the needed path length function. With the emission path length function $-\partial\Psi^{\pm}(l)/\partial l$ so defined, the desired mean emission length is obtained by applying the integral expression of equation (21). To verify the accuracy of the above procedure, plotted results for the mean emission length for the layer have been checked against values derived by applying the adding-doubling method to compute the volume emitted flux at the layer boundary. The derivative expression in equation (27) is then evaluated by finite differences and used to obtain the mean emission length. The layer results from this derivative method and those from the Monte Carlo method agree to within 2% and typically to within 1%. In addition, the Monte Carlo procedure used for the plotted values incorporates iterative techniques such that an uncertainty in the results can be specified. The 99% confidence interval for the plotted values is typically less than $\pm 5\%$ of the computed magnitude of the ratio of emission length to mean beam length. For a few worst case points, that percentage increases to about $\pm 7\%$.

There are several important trends to note in Figs. 3–11. At the limit of β_v approaching zero, note that the optically thin results discussed above apply. That is, the mean emission length $\langle l \rangle^{+}_{ve}$ approaches the local mean beam length. For the cylinder results, however, the surface area average mean beam length is used to scale $\langle l \rangle^{+}_{ve}$ rather than the local value arrived at by integration over the volume. Thus, for the cylinder results given, $\langle l \rangle^{+}_{ve}/L_{mb}$ does not limit to unity while $\langle l \rangle^{+}_{ve}/L_{mb,l}$ does. For example, consider the $r/D = 0$ results in Fig. 4 where $\langle l \rangle^{+}_{ve}/L_{mb} \cong 1.07$ at $\beta_v = 0$. Since it can be shown that $L_{mb}/L_{mb,l} \cong 1/1.07$, it is clear that $\langle l \rangle^{+}_{ve}$ approaches the local value of the mean beam length. For Figs. 5–11, the same type of behavior is expected at $\beta_v = 0$.

In addition, note that for the cylinder, the surface area average mean beam length L_{mb} is the same for all Figs. 4–11. Therefore, the overall magnitudes of the lengths for the various positions around the cylinder boundary can be compared and are generally typified by the magnitude at $\beta_v = 0$. Specifically, note that near

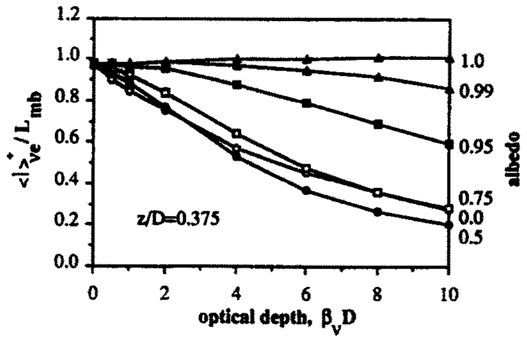

FIG. 9. Mean emission length at the surface of the cylinder at an axial location of $z/D = 0.375$.

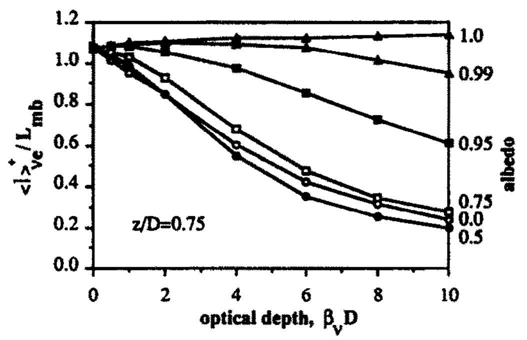

FIG. 10. Mean emission length at the surface of the cylinder at an axial location of $z/D = 0.75$.

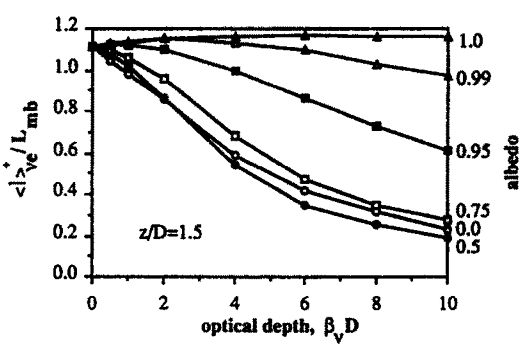

FIG. 11. Mean emission length at the surface of the cylinder at an axial location of $z/D = 1.5$.

the corner of the cylinder ($r/D = 0.499$ and $z/D = 0.001$) the magnitude of $\langle l \rangle^+_{ve}$ decreases markedly, being much larger near the axial center of the cylinder ($z/D = 1.5$) and the base center ($r/D = 0$).

To interpret the results in these figures, it is helpful to use equation (33), the important result from Section 4. That is, consider the original medium (a_v, σ_v, P_v) with a very small amount of a secondary gray absorber (a'_v) added. The emitted flux q^+_{ve} at a boundary location is, from equation (33)

$$q^+_{ve} \cong q^+_{ve}(a'_v = 0) - a'_v \langle l \rangle^+_{ve} [q^+_{ve}(a'_v = 0) - \pi i_{vb}(T)]. \quad (37)$$

As β_v approaches zero and albedo approaches unity (for arbitrary β_v), the emitted flux with no secondary absorber is zero. Thus

$$q^+_{ve} \cong \pi i_{vb}(T) a'_v \langle l \rangle^+_{ve} \quad (38)$$

which is valid for $a_v \to 0$, regardless of the value σ_v. The emitted flux is then directly proportional to the mean emission length. The decrease in $\langle l \rangle^+_{ve}$ at $\beta_v = 0$ and $\beta_v \neq 0$, $\omega_v = 1$ as the corner is approached is interpreted as a decrease in the local volume emitted flux. In the case of $\beta_v \neq 0$, $\omega_v \neq 1$, the decrease in the overall magnitude of $\langle l \rangle^+_{ve}$ is interpreted as a decrease in the local flux that originates as secondary absorber emission.

The behavior at $\omega_v = 1$ for the layer and cylinder can also be explained following the above arguments. Note that the mean emission length divided by the mean beam length is equal to a constant for the layer when albedo is equal to unity. Using the interpretation discussed above, the optically thin secondary absorber emits a given amount of energy that is not self-absorbed. Thus, all of the emitted energy must leave the medium—half through one boundary and half through the other, even after scattering has occurred. The emitted flux q^+_{ve} is then constant, regardless of the scattering coefficient, and because of equation (38), the mean emission length is constant.

For the cylinder, however, the argument as to the distribution of emitted energy at the boundary that is valid for the layer is inappropriate. That is, scattering in the cylinder is able to redistribute the energy around the boundary, increasing q^+_{ve} in some locations, decreasing the flux in others. Because of equation (38), $\langle l \rangle^+_{ve}$ behaves correspondingly. Therefore, at the axial center of the cylinder (Fig. 11), the mean emission length increases with increasing extinction coefficient when $\omega_v = 1$ while near the corner (Fig. 8) $\langle l \rangle^+_{ve}$ decreases with increasing β_v. This indicates that increased scattering serves to preferentially scatter the energy to the center of the cylinder rather than the corners.

The behavior of the mean emission length for $\omega_v \neq 1$ is a bit more complex. If the extinction coefficient is fixed and albedo is increased, the emission length varies nonmonotonically in some cases (see Figs. 3–6 and 9–11) and rather monotonically in others (see Figs. 7 and 8). The reason for this behavior is that by fixing β_v, increasing $\omega_v = \sigma_v/\beta_v = \sigma_v/(a_v + \sigma_v)$ causes σ_v to increase and a_v to correspondingly decrease. The mean emission length is expected to increase as a_v decreases for fixed σ_v. For fixed a_v, however, it is difficult to predict the influence of an increasing scattering coefficient since not only are path lengths altered, but also energy is redistributed around the boundary.

6. CONCLUSIONS

A technique for obtaining characteristic length scales for radiative transfer problems using the PPL approach is presented. The method is used to develop characteristic lengths for an arbitrarily shaped homogeneous medium which absorbs, emits, and scatters, yet the lengths may be computed without knowledge of the path length distribution functions. A characteristic emission length limits to the mean beam length for the optically thin media considered. The behavior of the emission length for media with larger values of absorption and scattering coefficient is studied. Finally, an example of how the emission length may be applied to radiation problems with a secondary absorber is provided.

Beyond the specific applications discussed above, the mean lengths have general interest as radiation length scales. Although length scales are used to a large extent in other heat transfer disciplines, developments in radiative transfer analysis are limited. Also of general interest is the methodology used in developing the lengths. Radiative intensities and mean lengths for intensity are easily obtained using direct extensions of the energy transfer formulas included here.

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SUR LES LONGUEURS CARACTERISTIQUES POUR LES MILIEUX ABSORBANTS, EMISSIFS ET DIFFUSANTS

Résumé—On présente les longueurs caractéristiques pour des milieux multidimensionnels absorbants, émissifs et diffusants. La longueur représentant les phénomènes d'émission est discutée en détail, alors que la longueur représentant les contributions de la réflexion et la transmission est dérivée simplement. Les longueurs caractéristiques sont développées à partir de l'approche photonique de la longueur de parcours, mais les longueurs sont calculables sans connaitre les fonctions de distribution des longueurs de parcours des photons. On présente des valeurs limites et la relation à la longueur moyenne du parcours. Des résultats numériques sont présentés et discutés pour des milieux plans et cylindriques avec diverses épaisseurs optiques et différents albedos.

ÜBER DIE CHARAKTERISTISCHE LÄNGE BEI ABSORBIERENDEN, EMITTIERENDEN UND STREUENDEN MEDIEN

Zusammenfassung—Die Entwicklung der charakteristischen Längen in mehrdimensionalen absorbierenden, emittierenden und streuenden Medien wird vorgestellt. Die charakteristische Länge für die Emission wird detailliert diskutiert, während die charakteristischen Längen für Reflexion und Transmissionsbeiträge einfach abgeleitet werden. Ausgangspunkt ist dabei eine Näherung für die Weglänge der Photonen. Die Längen können ohne Kenntnis der Verteilungsfunktion der Weglänge der Photonen (PPL) berechnet werden. Die Grenzwerte und die Abhängigkeit von der mittleren Weglänge werden dargestellt. Es werden numerische Ergebnisse für ebene und zylindrische Medien bei unterschiedlichen Werten für die optische Dicke und das Streu-Albedo angegeben und diskutiert.

ХАРАКТЕРНЫЕ ДЛИНЫ СОВОБОДНОГО ПРОБЕГА ЧАСТИЦ ДЛЯ ПОГЛОЩАЮЩИХ, ИЗЛУЧАЮЩИХ И РАССЕИВАЮЩИХ СРЕД

Аннотация—Представлены обобщенные методы определения характерных длин свободного пробега частиц для многомерных, поглощающих, излучающих и рассеивающих сред. Подобно обсуждается длина, характерная для явления излучения, а длина, характеризующая вклады отражения и пропускания, только определяется. Для определения характерных длин используется подход, основанный на длине свободного пробега фотона, однако они могут быть рассчитаны и без знания функций распределения длины свободного пробега фотона (ДСПФ). Представлены предельные значения и отношение к средней длине луча. Приводятся и обсуждаются численные результаты, полученные для плоских и цилиндрических сред для диапазонов оптической глубины и рассеивающего альбедо.